Part I: Linear Algebra (50 points)

True (T) or False (F) [20%]:
2 points for each correct answer, and −1 point for each wrong answer. Be careful.

1. If A and B are sysmmetryic 2×2 matrix, then AB is symmetric.
2. If an $n \times n$ matrix A is not symmetric, then $A^T A$ is not symmetric.
3. In general, the determinant of the sum of two matrices equals the sum of the determinants of the matrices.
4. If for some matrix A, and some vectors x, b, we have $Ax = b$, then b is in the span of the column vectors of A.
5. Let A and B be $n \times n$ matrices. Assume that $AB = I_n$. Then, $BA = I_n$.
6. The span of two non-zero vectors in R^2 is all of R^2.
7. If A and B are square matrices then $(A + B)^2 = A^2 + 2AB + B^2$.
8. The angle between two nonzero vectors in R^n is less than 90 degrees if and only if the dot product of the vectors is positive.
9. For any matrix A, the product AA^T is diagonalizable.
10. $T(x_1, x_2) = x_1^2 + x_2^2$ is a linear transformation.

Muliple Choice [30%]:
Each of the following questions has axactly one correct choice. 3 points for each correct choice, and −1 point for each wrong choice. Be careful.

1. Let A be a 3×4 matix and let B be a 4×3 matrix. Which of the following operations makes sense?

   (a) $A \times B$　(b) $A^T \times B$　(c) $A \times B^T$　(d) $A + B$　(e) $A - B$.

2. Let A be an invertible 5×5 matix. Which of the following statements is false?

   (a) The rank of A must equal 5.
   (b) The linear transformation assoiciated to A must be one-to-one and onto.
   (c) For every vector b in R^5, there must be exactly one solution to the equation $Ax = b$.
   (d) The reduced row echelon form of A must be the identity matrix.
   (e) Every row of A must contain a leaning 1.

3. A transformation $T : R^n \to R^m$ is linear if and only if

   (a) The image of T is a line.
   (b) T is one-to-one and onto.
   (c) There exists a matrix A such that $T(x) = Ax$ for all x in R^n.
   (d) One has $T(x + y) = T(x) + T(y)$ and $T(cx) = cT(x)$ for all vectors $x, y \in R^n$ and scalars c.
   (e) The graph of T takes the form $y = mx + c$

4. Given that $B = (1, -1, 0)^T, (0, 1, -1)^T, (2, 0, 2)^T$ is an order basis in R3, and vercor $v = (1, 1, 1)^T$ with respect to the standard basis, the representation of v with respect to B is

   (a) $(1/2, 1/2, 3/4)^T$
   (b) $(3/4, 1/2, -1/2)^T$
   (c) $(-1/2, 1/2, 3/4)^T$
   (d) $(1/2, -1/2, 1/2)^T$
   (e) $(-1/2, 1/2, -1/2)^T$

5. The first rwo of the inverse of the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ is

   (a) $\begin{bmatrix} 0 & -2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -3 & -6 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 6 & -3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 1 & 0 \end{bmatrix}$ (e) none.

6. The determinant of $\begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 5 & 4 \end{bmatrix}$ equals

   (a) -120 (b) -24 (c) 0 (d) 24 (e) 120.

7. For what real numbers c is the matrix $\begin{bmatrix} 1 & c \\ 0 & 2 \end{bmatrix}$ diagonalizable?

   (a) All nonzero real numbers c.
   (b) $c = 0$ only.
   (c) All real numbers except 1 and 2.
   (d) All real numbers c.
   (e) It is never diaonalizable.

8. The unique line $y = mx + b$ best approximating the data points (1, 2), (2, 3), (3, 5), in the sense of least squares is

   (a) $y = x + 3/2$.
   (b) $y = x + 4/3$.
   (c) $y = (3/2)x + 1/3$.
   (d) $y = (3/2)x + 1/6$.
   (e) The best approximating line is not uniquely determined by the given data.

9. What are the eigenvalues of $A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$?

   (a) $\lambda_1 = 2, \lambda_2 = -5$.
   (b) $\lambda_1 = -2, \lambda_2 = 5$.
   (c) $\lambda_1 = 2, \lambda_2 = 5$.
   (d) $\lambda_1 = -2, \lambda_2 = -5$.
   (e) Not available.

10. Which of the following statements is not true?

    (a) Similar matrices A and B have exactly the same eigenvectors.
    (b) Similar matrices A and B have exactly the same eigenvalues.
    (c) Similar matrices A and B have exactly the same determinant.
    (d) Similar matrices A and B have exactly the same characteristic polynomial.
    (e) None of the above.

Part II: Discrete Math (50 points)

1. (16%) For A = {5, $\varnothing$, {$\varnothing$}, 3, 4},
   (a) determine the number of nonempty subsets of A,
   (b) give an example of a relation R on A that is reflexsive,
   (c) give an example of a relation R on A that is reflexsive but not symmetric,
   (d) and find the power set of A.

2. (4%) Give an example of a <u>function</u> from N $\rightarrow$ N that is onto but not 1-1 (one to one).

3. (8%) The general solution of the recurrence relation $a_{n+2} + b_1 a_{n+1} + b_2 a_n = b_3 n + b_4$, $n \geq 0$, with b_i constant for $1 \leq i \leq 4$, is $c_1 2^n + c_2 3^n + n - 7$. Find b_i for each $1 \leq i \leq 4$.

4. (4%) Use the definition of big-oh to prove $(6n + 4n^4 - 14) / (n^2 - 3)$ is $O(n^2)$. You must state the "witness" C and k clearly.

5. (12%) Translate the following statements using predicates, quantifiers, logical connectives, and mathematical operators.
   (a) There is no <u>dog</u> that can <u>talk</u>. [domain: all animals]
   (b) All your <u>friends</u> are <u>perfect</u>. [domain: all people]
   (c) The average of two positive real numbers is positive. [domain: real numbers]

6. (6%) Show that these two statements about integer n are equivalent: (i) $3n + 2$ is even (ii) $n + 5$

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