

# 大同大學 九十 學年度研究所碩士班入學考試試題

考試科目：基本數學

所別：資訊工程研究所

第 1/3 頁

註：本次考試 不可以參考自己的書籍及筆記； 不可以使用字典； 不可以使用計算器。

This examination paper contains three parts: (A) Discrete Mathematics; (B) Linear Algebra; and (C) Probability. Please choose any two parts to test. If all parts are chosen, the answers of the part that appear as the 3<sup>rd</sup> order in the answer sheet will be ignored.

$$(p \vee q) \rightarrow r \leftrightarrow$$

$$\neg(p \vee q) \vee r \rightarrow$$

$$r \vee \neg(p \vee q)$$

## Part (A): Discrete Mathematics

1. Prove that if  $n$  and  $k$  are positive integers with  $n = 3k$ , then  $n!/3^k$  is an integer. (5%)

$$\frac{n!}{3^k} = \frac{3k!}{3^k} = \frac{3k(3k-1)!}{3^k}$$

2. How many arrangements of the letters in MISSISSIPPI have no consecutive S's? (5%)

3. Let  $p, q$  be primitive statements for which the implication  $p \rightarrow q$  is false. Determine the truth values for each of the following. (a)  $p \wedge q$  (b)  $\neg p \vee q$  (c)  $q \rightarrow p$  (d)  $\neg q \rightarrow \neg p$ . (5%)

$$p \rightarrow q \text{ is false} \Rightarrow p = T, q = F$$

M I I I P P I

4. Use the substitution rules to verify that each of the following is a tautology. (Here  $p, q,$  and  $r$  are primitive statements.) (a)  $[p \vee (q \wedge r)] \vee \neg[p \vee (q \wedge r)]$  (b)  $[(p \vee q) \rightarrow r] \leftrightarrow [\neg r \rightarrow \neg(p \vee q)]$ . (5%)

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40320$$

5. Let  $p(x)$  be the open statement " $x^2 = 2x$ ," where the universe comprises all integers. Determine whether each of the following statements is true or false. (a)  $p(1)$  (b)  $p(-2)$  (c)  $\exists x p(x)$  (d)  $\forall x p(x)$ . (5%)

$$\binom{8}{4} = \frac{8!}{4!4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70$$

6. (a) Determine the sets  $A, B$  where  $A - B = \{1, 3, 7, 11\}$ , and  $B - A = \{2, 6, 8\}$ , and  $A \cap B = \{4, 9\}$ . (b) Determine the sets  $C, D$  where  $C - D = \{1, 2, 4\}$ , and  $D - C = \{7, 8\}$ , and  $C \cup D = \{1, 2, 4, 5, 7, 8, 9\}$ . (5%)

$$= \frac{8 \cdot 7 \cdot 6 \cdot 5}{4} = 49 \cdot 15 = 735$$

7. Prove the following by mathematical induction. (5%)

25. (a)  $\sum_{i=1}^n \frac{1}{i(i+1)} = \frac{n}{n+1}$  (b)  $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \dots + n(n+2) = (n)(n+1)(2n+7)/6$



$$\frac{150}{49} = 3 \frac{5}{49}$$

8. Determine whether or not each of the following relations is a function. If a relation is a function, find its range. (5%)

$$[p \vee (q \wedge r)] \vee \neg[p \vee (q \wedge r)]$$

- (a)  $\{(x, y) \mid x, y \in \mathbf{Z}, y = x^2 + 7\}$ , a relation from  $\mathbf{Z}$  to  $\mathbf{Z}$ .  
 (b)  $\{(x, y) \mid x, y \in \mathbf{R}, y^2 = x\}$ , a relation from  $\mathbf{R}$  to  $\mathbf{R}$ .  
 (c)  $\{(x, y) \mid x, y \in \mathbf{R}, y = 3x + 1\}$ , a relation from  $\mathbf{R}$  to  $\mathbf{R}$ .  
 (d)  $\{(x, y) \mid x, y \in \mathbf{Q}, x^2 + y^2 = 1\}$ , a relation from  $\mathbf{Q}$  to  $\mathbf{Q}$ .  
 (e)  $\mathcal{R}$  is a relation from  $A$  to  $B$  where  $|A|=5, |B|=6$ , and  $|\mathcal{R}|=6$ .

$$R \vee \neg R$$

9. Let  $M = (S, I, O, N, T)$  be a finite state machine where  $S = \{s_0, s_1, s_2, s_3\}, I = \{a, b, c\}, O = \{0, 1\}$ , and  $N, T$  are determined by the following table.

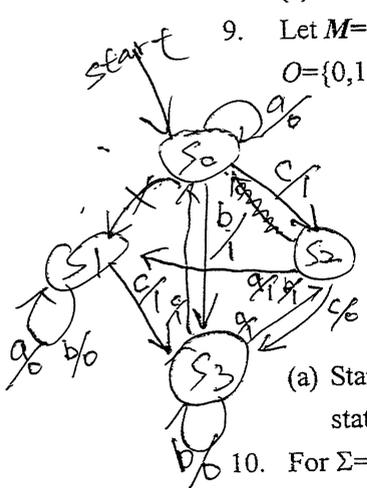
	N			T		
	a	b	c	a	b	c
$s_0$	$s_0$	$s_3$	$s_2$	0	1	1
$s_1$	$s_1$	$s_1$	$s_3$	0	0	1
$s_2$	$s_1$	$s_1$	$s_3$	1	1	0
$s_3$	$s_2$	$s_3$	$s_0$	1	0	1

$$\sum_{i=1}^{n-1} \frac{1}{2^i(2^i+1)} = \frac{n-1}{2^n-1}$$

$$\sum_{i=1}^n \frac{1}{2^i(2^i+1)}$$

$$\frac{1}{n(n+1)} + \frac{n-1}{n} = \frac{1 + (n-1)(n+1)}{n(n+1)} = \frac{1 + n^2 - 1}{n(n+1)} = \frac{n^2}{n(n+1)} = \frac{n}{n+1}$$

- (a) Starting at  $s_0$ , what is the output for the input string  $abbccc$ ? (b) Draw the state diagram for this finite state machine. (5%)



10. For  $\Sigma = \{0, 1\}$ , let  $A \subseteq \Sigma^*$  be the language defined recursively as follows: (1) The symbols 0, 1 are both in  $A$  — this is the base for our definition; and (2) For each word  $x$  in  $A$ , the word  $0x1$  is also in  $A$  — this constitutes the recursive process. (a) Find four different words — two of length 3 and two of length 5 — in  $A$ . (b) Use the given definition to show that 0001111 is in  $A$ . (5%)

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## Part (B): Linear Algebra

1. Consider the following matrix A:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad \rightarrow$$

$$\begin{bmatrix} -1 & -1 & 0 \\ \cancel{-1} & \cancel{2} & \cancel{-1} \\ 0 & -1 & -1 \end{bmatrix} \quad \begin{array}{l} -x_1 = x_2 \\ -x_1 = x_3 \\ -x_2 = x_3 \end{array}$$

$$x_1 = -x_2 = x_3$$

- Determine the rank of A. (5%)
- Determine the eigenvalues and the corresponding eigenvectors of A. (12%)
- Does every singular matrix have *zero* as one of its eigenvalues? (3%) If so, explain why. (4%)

2. Consider the problem of solving a linear system  $Ax = b$ :

$$\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Illustrate (畫出) the column space of A on your answer book. (6%). Show that the above system is inconsistent. (2%).
- Use the least square method to get an optimal solution for this problem. (Hint: obtain pseudo inverse of A). (12%)
- In your previous illustration, show that the optimal solution is the orthogonal projection of b in the column space. (6%)

$$\begin{array}{l} -x_3 \\ -x_1 + x_2 + x_3 = 0 \\ -x_1 = x_3 \end{array} \quad \begin{array}{l} -x_1 - x_2 = 0 \\ +x_2 + x_3 = 0 \\ \rightarrow x_3 = 0 \\ x_3 = 0 \end{array}$$

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## Part (C): Probability

### Multiple Choice:

In the following, there are 10 problems. Each of them has only one correct choice and scores 5 points. Making a wrong choice on a problem, you will get -2 points for that problem. The minimum score of this test is zero.

Answer problem 1 and 2 using the following problem setting:

Let  $X$  be a random variable with pdf  $f(x) = kx^{k-1}, 0 < x < 1$  and  $E[X] = 2/3$ .

1. The value of  $k = ?$  (a) 0.5; (b) 1; (c) 1.5; (d) 2.
2. The value of  $Var[X] = ?$  (a) 3/18; (b) 5/18; (c) 7/18; (d) 9/18.

Answer problem 3 and 4 using the following problem setting:

Let  $A$  and  $B$  be two events with  $P(A) = 0.4$  and  $P(B) = 0.5$ .

3. If  $A$  and  $B$  are independent, then  $P(A \cup B) = ?$  (a) 0.9; (b) 0.8; (c) 0.7; (d) 0.5.
4. If  $A \cup B = 0.6$ , then  $P(A' \cup B) = ?$  (where  $A'$  denotes the complement of event  $A$ ) (a) 0.9; (b) 0.8; (c) 0.7; (d) 0.5.

Answer problem 5 and 6 using the following problem setting:

Let  $X$  be a random variable with pdf  $f(x) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}, x > 0$ .

5. Suppose that  $X$  is used to model the life time of a component. Then, the corresponding failure rate  $h(x) = ?$  (a)  $\frac{x}{\sigma^2}$ ; (b)  $\frac{x^2}{\sigma^2}$ ; (c)  $\frac{x}{2\sigma^2}$ ; (d)  $\frac{x^2}{2\sigma^2}$ .
6. Let  $Y = g(X)$  be a function of  $X$ . Then, which of the following functions makes  $Y$  to be uniformly distributed in  $[0, 1]$ ? (a)  $g(X) = 1 - \frac{1}{2}\sigma e^{-X^2/2\sigma^2}$ ; (b)  $g(X) = 1 - \frac{X}{2\sigma^2} e^{-X^2/2\sigma^2}$ ; (c)  $g(X) = 1 - e^{-X^2/2\sigma^2}$ ; (d)  $g(X) = 1 - X e^{-X^2/2\sigma^2}$ .
7.  $\frac{1}{4!} \int_0^3 2^5 x^4 e^{-2x} dx = ?$  (a)  $\sum_{k=0}^4 \frac{6^k e^{-6}}{k!}$ ; (b)  $1 - \sum_{k=0}^4 \frac{6^k e^{-6}}{k!}$ ; (c)  $\sum_{k=0}^5 \frac{5^k e^{-5}}{k!}$ ; (d)  $1 - \sum_{k=0}^5 \frac{5^k e^{-5}}{k!}$ .
8. Suppose that  $n$  independent trials, each of which results in any outcomes of 0, 1, or 2, with probabilities 0.3, 0.5, and 0.2 respectively, are performed. Let  $A$  denote the event that both outcome 1 and outcome 2 occur at least once. Then, (a)  $0.82 < P(A) \leq 0.84$ ; (b)  $0.84 < P(A) \leq 0.86$ ; (c)  $0.86 < P(A) \leq 0.88$ ; (d)  $0.88 < P(A) \leq 0.90$ .
9. A box contains 3 red balls and 2 black balls. Now, two balls are selected without replacement. Let  $X$  and  $Y$  denote the numbers of red balls and black balls, respectively. Then, the correlation coefficient  $\rho_{X,Y} = ?$  (a) -1; (b) 0; (c) 1; (d)  $\frac{2}{3}$ .
10. The joint probability function of  $X$  and  $Y$  is given by  $f(x,y) = e^{-(x+y)}, 0 \leq x < \infty, 0 \leq y < \infty$ . Then,  $P(X+Y > 1) = ?$  (a)  $e^{-1} + e^{-2}$ ; (b)  $e^{-1} - e^{-2}$ ; (c)  $2e^{-1}$ ; (d)  $e^{-1}$ .