大同大學 九十四 學年度研究所碩士班入學考試試題

考試科目：基本數學　　所別：資訊工程研究所　　

註：本次考試 不可以參考自己的書籍及筆記； 不可以使用字典； 不可以使用計算器。

本試卷計兩部分：**PART I: 離散數學 (Discrete Math)**，**PART II: 線性代數 (Linear Algebra)**，各五十分，作答時，請清楚將各部分標示於答案卷內，以免影響成績。與答案無關之內容切勿書寫於答案卷內，否則以零分計。

PART I: Discrete Math

1. Find a proposition with three variables p, q, and r that is true when exactly one of the three variables is true, and false otherwise. (3%)
2. Prove or disprove: if p is prime and $p > 2$, then $p^2 + 4$ is prime. (3%)
3. Find the prime factorization of $8,827$. (3%)
4. Give an example of a function $f : Z \to N$ that is both 1-1 and onto N. (3%)
5. Prove or disprove: If $f(n) = n^2 - n + 17$, then $f(n)$ is prime for all positive integers n. (3%)
6. Prove or disprove: if A, B, and C are sets, then $A - (B \cap C) \overset{?}{=} (A - B) \cap (A - C)$. (3%)
7. What is the most efficient way to multiply the matrices A_1, A_2, A_3 of sizes 20×5, 5×50, 50×5? (3%)
8. Express $gcd(84, 18)$ as a linear combination of 18 and 84. (3%)
9. Let $A = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$. Find A^n where n is a positive integer. (4%)
10. Find the coefficient of x^8 in the expansion of $(x^2 + 2)^{13}$. (3%)
11. Find the sum $1 - 1/2 + 1/4 - 1/8 + 1/16 - \cdots$. (3%)
12. How many permutations of the seven letters A, B, C, D, E, F, G have the two vowels before the five consonants? (3%)
13. A club with 20 women and 17 men needs to choose three different members to be president, vice president, and treasurer.
    (a) In how many ways is this possible? (2%)
    (b) In how many ways is this possible if women will be chosen as president and vice president and a man as treasurer? (2%)

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14. Suppose $|A| = 4$ and $|B| = 10$. Find the number of 1-1 functions $f : A \to B$. (3%)

15. Using the ordinary alphabet and allowing repeated letters, find the number of words of length 8 that begin with L or end with R. (3%)

16. Suppose $|A| = n$. Find the number of binary relations on A. (3%)

PART II: Linear Algebra

I. (20%) True/False Problems: Indicate true (○) or false (×) for each below statement. (2 points each, and with penalty of -0.5 point for each incorrectness.)

1. Square matrices A and B are invertible if and only if AB is invertible.
2. If V is spanned by k vectors, $dim(V) \le k$.
3. If U is an $m \times n$ matrix with orthonormal columns, then $U^T U = I$.
4. If U is an $m \times n$ matrix with orthonormal columns, then $UU^T = I$.
5. Similar matrices have the same eigenvalues.
6. Similar matrices have the same eigenvectors.
7. If A and B are similar matrices, and $Av = 0$, then $Bv = 0$ as well.
8. Suppose A and B are 5×5 matrices and that $rank(A) = 3$ and $nullity(B) = 2$. Then $rank(AB)$ is at least 3.
9. Let A and B be $n \times n$ matrices, with $AB = BA$. Then $A^3 B = BA^3$.
10. If V is an inner product space and $u, v \in V$ satisfy $\langle u, v \rangle = 0$, then either $u = 0$ or $v = 0$.

II. (30%) Multiple-Choice Problems: Provide a correct choice for each of the following problems. (3 points each, and with penalty of -1 point for each wrong choice.)

1. Let V be the set of all positive real numbers and W be the set of infinite sequences of real numbers satisfying $(a_1, a_2, \ldots, a_n, \ldots) \in W$ if and only if $a_n = 2a_{n-1} - 3a_{n-2}$, for all $n > 2$. Operations on V and W are defined as follows:

$$\begin{aligned} V: \quad & v_1 + v_2 = v_1 v_2, \text{ for all } v_1, v_2 \in V \\ & kv = v^k, \text{ for all } k \in \mathbb{R}, v \in V \\ W: \quad & \text{For all } (a_1, a_2, \ldots, a_n, \ldots), (b_1, b_2, \ldots, b_n, \ldots) \in W \\ & (a_1, a_2, \ldots, a_n, \ldots) + (b_1, b_2, \ldots, b_n, \ldots) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n, \ldots) \\ & k(a_1, a_2, \ldots, a_n, \ldots) = (ka_1, ka_2, \ldots, ka_n, \ldots) \text{ for all } k \in \mathbb{R} \end{aligned}$$

With these definitions, which of the following statements is true: (a) V is a vector space, W is not; (b) W is a vector space, V is not; (c) both V and W are vector spaces; (d) neither V nor W is a vector space; (e) none of these.

2. Which of the following functions are linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$:

i) $R_\alpha(v)$ is a rotation of any $v \in \mathbb{R}^3$ around the axis $x = y = z$ by angle α.

ii) $S(v) = 2v$, for all $v \in \mathbb{R}^3$.

iii) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - 3y + z \\ y + z \\ z \end{pmatrix}$.

iv) $U\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - 3y + z \\ y + z \\ 1 + z \end{pmatrix}$.

(a) i, iii; (b) ii; (c) i, ii, iii; (d) i, iii, iv; (e) i, ii, iii, iv.

3. Consider the following sets of vectors:

$$\begin{aligned} S_1 &= \{(1,0,1)^T, (2,1,1)^T, (1,1,0)^T\} \\ S_2 &= \{(2,1,0)^T, (3,-2,0)^T, (0,1,0)^T\} \\ S_3 &= \{(1,-1,0)^T, (0,1,-1)^T, (2,0,2)^T\} \end{aligned}$$

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Of these sets, those which are bases of $\mathbb{R}^3$ are: (a) All of them; (b) S_1, S_3; (c) S_2; (d) S_3; (e) none of them.

4. Let α and β be the eigenvalues of the 2×2 matrix $A = \begin{bmatrix} 14 & -2 \\ -2 & 11 \end{bmatrix}$. Then, $(\alpha - \beta)^2 =$ (a) 16; (b) 25; (c) 36; (d) 49; (e) 64.

5. If $Ax = b$ has a unique solution, and A is 4×4 and b is 4×1, which of the following statements is False? (a) $det(A) = 0$; (b) rank of A = rank of $[A|B]$; (c) $Ax = 0$ has a non-trivial solution; (d) $Ax = 0$ has a unique solution; (e) $x = A^{-1}b$.

6. If A, B, C are square invertible matrices and $ABC = I$, then B^{-1} is: (a) $A^{-1}C^{-1}$; (b) $C^{-1}A^{-1}$; (c) CA; (d) AC; (e) none of these.

7. Which of the following forms an orthonormal basis for $\mathbb{R}^3$?
(a) $\left\{ \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right\}$; (b) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\}$;
(c) $\left\{ \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix}, \begin{pmatrix} -\frac{2}{\sqrt{3}} \\ -\frac{2}{\sqrt{3}} \\ -\frac{2}{\sqrt{3}} \end{pmatrix}; \right\}$; (d) $\left\{ \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix} \right\}$;
(e) none of these.

8. If $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$, and $det(A) = 10$, then the determinant of $\begin{bmatrix} d_1 & d_2 & d_3 & d_4 \\ 2b_1 & 2b_2 & 2b_3 & 2b_4 \\ 3a_1 & 3a_2 & 3a_3 & 3a_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}$ is
(a) $(2^4)(3^4)(10)$; (b) -60; (c) 60; (d) $-(2^4)(3^4)(10)$; (e) none of these.

9. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. Which of the following is similar to A?
(a) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$; (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; (c) $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$; (d) $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$; (e) none of these.

10. The vectors v_1, v_2 and v_3 form an orthonormal basis for $\mathbb{R}^3$ where $v_1 = (0, 0, -1)^T$, $v_2 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)^T$, and $v_3 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)^T$. The coordinates of the vector $w = (1, -1, 1)^T$ with respect to the basis vectors v_1, v_2 and v_3 of $\mathbb{R}^3$ are: (a) $-1, 0, -\frac{2}{\sqrt{2}}$; (b) $1, -\frac{2}{\sqrt{2}}, 0$; (c) $1, \frac{2}{\sqrt{2}}, 0$; (d) $-1, -\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}$; (e) none of these.