大同大學 96 學年度研究所碩士班入學考試試題

考試科目:基本數學　　所別:資訊工程研究所　　

註:本次考試 不可以參考自己的書籍及筆記； 不可以使用字典； 不可以使用計算器。

Part I: 線性代數

I. (30%) True/False Problems:

Indicate true (◯) or false (×) for each below statement.

(2 points each, and with penalty of −1 point for each wrong answer.)

1. The empty set is a vector space over any field.
2. Any three non-zero vectors span $\mathbb{R}^3$.
3. The vector space $\mathbb{R}^3$ has a basis containing the vector $(1, 2, 3)$.
4. If $V = \text{span}\{v_1, \ldots, v_n\}$, then $dim(V) \leq n$.
5. If A and B are square matrices and AB is the identity matrix, then BA is also the identity matrix.
6. If A, B are row equivalent matrices, then $det(A) = det(B)$.
7. If a system of linear equations has two different solutions, then it must have infinitely many solutions.
8. A set of three vectors in $\mathbb{R}^2$ can be linearly independent.
9. Two eigenvectors with the same eigenvalue are always linearly independent?
10. Let a, b, c, d be scalars such that $ad - bc = 1$. Then, $(x, y) = (d - b, a - c)$ is a solution to the system
$$\begin{aligned} ax + by &= 1 \\ cx + dy &= 1. \end{aligned}$$
11. If A is an $n \times n$ matrix, then $det(A) = det(A^T)$.
12. An $n \times n$ matrix with n different real eigenvalues is diagonalizable.
13. If A is an $n \times n$ matrix with fewer than n distinct eigenvalues, then A is not diagonalizable.
14. Two similar matrices share the same eigenvectors.
15. If A is a non-singular 4×4 matrix, then $det(2A) = 2det(A)$.

II. (20%) Multiple-Choice Problems:

Each of the following problems has only one correct choice. Provide the correct choice for each of them.

(2 points each, without penalty on wrong choice.)

1. Which of $U = \{(x, y, x - y) : x, y \in \mathbb{R}\}$, $V = \{(x, y, x + y) : x, y \in \mathbb{R}\}$ and $UW = \{(x, y, xy) : x, y \in \mathbb{R}\}$ are subspaces of $\mathbb{R}^3$?
   (a) U only
   (b) V only
   (c) W only
   (d) U and V only
   (e) V and W only
   (f) W and U only
   (g) All of U, V and W.

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2. Given $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 4 & 1 \\ 3 & 4 \end{bmatrix}$, $(AB)^{-1} =$?
   (a) $\begin{bmatrix} 1 & -5 \\ 2 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & -\frac{3}{2} \\ \frac{5}{2} & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix}$ (e) $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -5 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} \frac{1}{2} & -\frac{5}{2} \\ 2 & 3 \end{bmatrix}$
   (g) $\begin{bmatrix} -\frac{1}{2} & \frac{5}{2} \\ 2 & -3 \end{bmatrix}$.

3. The coefficient matrix A in a homogeneous system of 12 equations in 16 unknowns is known to have rank 6. How many free parameters are there in the solution?
   (a) 10 (b) 6 (c) 4 (d) 16 (e) 12 (f) 22 (g) none.

4. For a nonhomogeneous system of 10 equations in 12 unknowns, which one of the suggested combinations of answers to the following questions is correct.

   - Can the system be inconsistent?
   - Can the system have infinitely many solutions?
   - Can the system have only one solution?

   (a) Yes, Yes, Yes
   (b) No, No, No
   (c) No, No, Yes
   (d) Yes, Yes, No
   (e) No, No, No
   (f) No, Yes, Yes
   (g) Yes, No, No.

5. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$, then $\begin{vmatrix} 4g & a & d-2a \\ 4h & b & e-2b \\ 4i & c & f-2c \end{vmatrix}$ =?
   (a) 6 (b) −12 (c) −24 (d) 12 (e) 24 (f) −6 (g) 8.

6. Given the matrices $A = \begin{bmatrix} 7 & 5 & 1 \\ 2 & 0 & 4 \\ 3 & 0 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & -1 \\ 1 & 2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 7 & 6 \\ 0 & 1 & 1 \\ 0 & 2 & -5 \end{bmatrix}$, which one of the following statements is true?
   (a) only A is invertible
   (b) only B is invertible
   (c) only C is invertible
   (d) A and B are both invertible
   (e) B and C are both invertible
   (f) A and C are both invertible
   (g) none is invertible.

7. If A is a 3×3 matrix with an eigenvalue 2 whose associated eigenvector is $\begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$, then $A^2 = A \cdot A$ has an eigenvalue λ whose associated eigenvector v is:
   (a) $\lambda = 4, v = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ (b) $\lambda = 4, v = \begin{bmatrix} 3^2 \\ 6^2 \\ 3^2 \end{bmatrix}$ (c) $\lambda = 2, v = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ (d) $\lambda = 2, v = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$
   (e) $\lambda = \sqrt{2}, v = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$ (f) $\lambda = \sqrt{2}, v = \begin{bmatrix} 3^2 \\ 6^2 \\ 3^2 \end{bmatrix}$ (g) none of the above.

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8. Let W be a subspace of $\mathbb{R}^4$ with an orthogonal basis $\left\{ v_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix}, v_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \\ -3 \end{bmatrix} \right\}$.

We can write $u = \begin{bmatrix} 15 \\ -1 \\ -2 \\ 4 \end{bmatrix}$ as $u = u_1 + u_2$ where $u_1 \in W$, $u_2 \in W^{\perp}$. What it u_2?

(a) $u_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ (b) $u_2 = \begin{bmatrix} -8 \\ 0 \\ 7 \\ -3 \end{bmatrix}$ (c) $u_2 = \begin{bmatrix} 11 \\ 3 \\ -7 \\ 3 \end{bmatrix}$ (d) $u_2 = \begin{bmatrix} 5 \\ -3 \\ 7 \\ -3 \end{bmatrix}$ (e) $u_2 = \begin{bmatrix} 11 \\ -5 \\ 14 \\ -6 \end{bmatrix}$ (f) $u_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(g) none of the above.

9. Compute $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{2007}$.

(a) $\begin{bmatrix} 1 & 0 & 0 & -3^{2007} \\ 0 & 1 & 3^{2007} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & 3^{2007} \\ 0 & 1 & -3^{2007} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & -6021 & 0 \\ 0 & 1 & 0 & 6021 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 & 6021 \\ 0 & 1 & -6021 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & 0 & -6021 \\ 0 & 1 & 6021 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(g) none of the above.

10. The matrix $A = \begin{bmatrix} 8 & -3 \\ 18 & 7 \end{bmatrix}$ is diagonalizable. Which of the following could be $P^{-1}AP$ for some invertible matrix P?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$ (f) $\begin{bmatrix} 4 & 0 \\ 0 & -5 \end{bmatrix}$ (g) $\begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$.

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Part II: 離散數學　　〈接前頁〉

1. Determine whether $(p \to (q \vee r)) \to ((p \to r) \vee (p \to q))$ is a tautology. [4%]

2. How many ways are there to assign six different jobs to three different employees if the hardest job is assigned to the most experienced employee and the easiest job is assigned to the least experienced employee? [6%]

3. What is wrong with this "proof"? [4%]
Theorem: $a^n = 1$ for all nonnegative integers n, where a is a nonzero real number.
Basic step: $a^0 = 1$ is true by the definition of a^0.
Inductive step: Assume that $a^j = 1$ for all nonnegative integers j with $j \le n$. Then note that

$$a^{k+1} = \frac{a^k \cdot a^k}{a^{k-1}} = \frac{1 \cdot 1}{1} = 1.$$

4. Use mathematical induction to show that 21 divides $4^{n+1} + 5^{2n-1}$ whenever n is a positive integer. [6%]

5. (a) If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Explain your answer. [4%]
(b) If f and $f \circ g$ are onto, does it follow that g is onto? Explain your answer. [4%]

6 Find the number of solutions of $x_1 + 2x_2 + x_3 = 17$ where x_i's are nonnegative integers with $2 \le x_1 \le 5, 3 \le x_2 \le 6, 4 \le x_3 \le 7$ [8%]

7. Solve the following recurrence relation [8%].

$$a_n = 5a_{n-1} - 6a_{n-2} + 2^n, a_0 = 1, a_1 = 1$$

8. The Computer Science Department has 5 committees that meet once a month. How many different meeting times must be used to guarantee that no one is scheduled to be at 2 meetings at the same time? The committees and their members are C_1 = {Adams, Brooks, Casey}, C_2 = {Brooks, Casey, Frank}, C_3 = {Brooks, Daly, Ellen}, C_4 = {Ellen, Frank}, C_5 = {Brooks, Casey, Daly} [6%].