線性代數（計五十分）

True or false (answer the questions; <u>*no*</u> explanation required) [14%]

1. If the rows of an *m-by-n* matrix A are linearly independent, then $Ax = b$ is always solvable.
2. If the rows of an *m-by-n* matrix A are linearly independent, then the solution of $Ax = b$, if it exists, is always unique.
3. If the columns of an *m-by-n* matrix A are linearly independent, then $Ax = b$ is always solvable.
4. For *n-by-n* real symmetric matrices A and B, AB and BA always have the same eigenvalues.
5. For *n-by-n* matrices A and B with B invertible, AB and BA always have the same eigenvalues.
6. Two diagonalizable matrices A and B with the same eigenvalues and eigenvectors must be the same.
7. If the columns of a matrix are dependent, so are the rows of the matrix.

8. [6%] The matrix A and vector b are

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & 1 & 0 & 3 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

Find the complete solution to $Ax = b$.

9. Find the determinant of this N matrix. [3%]

$$N = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 2 & 1 & 0 & 3 \\ 3 & 0 & 1 & 2 \\ 4 & 0 & 0 & 1 \end{bmatrix}$$

What is the rank of $N - I$? [2%] Find all four eigenvalues of N. [5%]

10. The (real) matrix A is

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & x & 3 \\ 2 & 3 & 6 \end{bmatrix}$$

For which values of x is A positive definite? [5%]
For which values of x is A^2 positive definite? [5%]

11. [10%] This problem uses least square to find the plane $C + Dx + Ey = b$ that *best fits* these 4 points.

x	y	b
0	0	2
1	1	1
1	-1	0
-2	0	1

Write down the equations <u>and</u> solve for the best (least square) plane.

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離散數學（計五十分）

True (T) or False (F):
2 points for each correct answer, and −1 point for each wrong answer. Be careful.

1. If A, B, and C are any three finite sets, and $A \cap B \cap C = \varnothing$, then $|A \cup B \cup C| \le |A| + |B| + |C|$.
2. No function can be both $O(n^2)$ and $O(n^3)$.
3. If a language is regular then it is finite.
4. Language $a^n b^n$ is regular.
5. There is no known algorithm to decide if a graph has a Hamiltonian cycle.
6. If a Boolean formula is tautological then it is satisfiable.
7. $(p \vee q) \to (\neg p \to q)$ is a tautology.
8. The set of rational numbers Q is countably infinite.
9. A connected graph G is a tree if and only if G has fewer vertices than edges.
10. $\binom{100}{0} + \binom{100}{2} + \binom{100}{4} + \cdots + \binom{100}{100} = \binom{100}{1} + \binom{100}{3} + \binom{100}{5} + \cdots + \binom{100}{99}$.

Muliple Choice:
Each of the following questions has axactly one correct choice. 2 points for each correct answer, and −0.5 point for each wrong answer. Be careful.

1. A biased coin with $P(H) = 1/10$ is tossed 3 times. What is the probability of getting exactly one H?
   - (a) $6\left(\dfrac{1}{10}\right)\left(\dfrac{9}{10}\right)^2$;
   - (b) $6\left(\dfrac{9}{10}\right)\left(\dfrac{1}{10}\right)^2$;
   - (c) $3\left(\dfrac{1}{10}\right)\left(\dfrac{9}{10}\right)^2$;
   - (d) $3\left(\dfrac{9}{10}\right)\left(\dfrac{1}{10}\right)^2$;
   - (e) none of the above.

2. Determine how many strings can be formed by rearranging the letters $ABCDE$ so that the letters ACE are together but may be in any order.
   - (a) $3! \cdot 3!$;;
   - (b) $5!/3!$;
   - (c) $5!$;
   - (d) $5! \cdot 3!$;
   - (e) none of the above.

3. Find the coefficient of $x^6 y^6$ in $(2x - y)^{12}$.
   - (a) -59136;
   - (b) 59136;
   - (c) $\binom{12}{6}$;
   - (d) $-\binom{12}{6}$;
   - (e) none of the above.

4. Assume that p and r are false and q is true, and let F stand for false while T stand for true. Find the truth values of the following four propositions:

$$\neg(p \to q), \quad (p \to q) \wedge (q \to r), \quad (p \to q) \to r, \quad p \to (q \to r)$$

   (a) TFTT;
   (b) FFFT;
   (c) FTTF;
   (d) TFFT;
   (e) none of the above.

5. Suppose that we have two matrix $A = (a_{ij})_{m \times p}$ which is m£p and a matrix $B = (b_{ij})_{p \times n}$. The following is an algorithm for calculating $C = AB$.

```
for i = 1 to m
    for j = 1 to n
        cij = 0
        for k = 1 to p
            cij = cij + aikbkj
        endfor
    endfor
endfor
```

   (a) The algorithm is $O(p^3)$;
   (b) The algorithm is $O(mnp)$;
   (c) The algorithm is $O(mnp^2)$;
   (d) The algorithm is $O(m^2np^2)$;
   (e) none of the above.

6. Define the function $f(n)$ by $f(1) = 1$ and $f(n) = n + f(n-1)$ for $n > 1$. What is a non-inductive representation for $f(n)$?
   (a) $f(n) = n(n+1)/2$;
   (b) $f(n) = n+1$;
   (c) $f(n) = n$;
   (d) $f(n) = 2n-1$;
   (e) none of the above.

7. Let $|A| = 3$ and $|B| = 4$. The number of function $f : A \to B$ is:
   (a) 3^4;
   (b) 4^3;
   (c) 2^7;
   (d) 2^{12};
   (e) none of the above.

8. Let $|A| = 3$ and $|B| = 4$. The number of relations from A to B is:
   (a) 3^4;
   (b) 4^3;
   (c) 2^7;
   (d) 2^{12};
   (e) none of the above.

9. Which of the following is **not** in the set denoted by regular expression (a b*c*)*
   (a) abc;
   (b) bcc;
   (c) acc;
   (d) abcac;
   (e) none of the above.

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註：本次考試　不可以參考自己的書籍及筆記；　不可以使用字典；　不可以使用計算器。

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10. Random variable is a:
    (a) variable;
    (b) probability;
    (c) function;
    (d) data value;
    (e) none of the above.

11. Which of the following distriutions is not discrete:
    (a) Gamma disrtibution;
    (b) Poison distribution;
    (c) Geometric distribution;
    (d) binomial distribution;
    (e) none of the above.

12. Which of the following distriutions is memoryless:
    (a) Gamma disrtibution;
    (b) Poison distribution;
    (c) Geometric distribution;
    (d) binomial distribution;
    (e) none of the above.

13. $1 + .1 + .1^2 + .1^3 + \cdots 1 =$
   (a) 11/9
    (b) 10/8;
    (c) 11/9;
    (d) 10/9;
    (e) none of the above.

14. The probability about a binomially disributed random variable can be approximated using a Poison distribution if
    (a) n is very small and p very small;
    (b) n is very large and p very small;
    (c) n is very small and p very large;;
    (d) n is very large and p very large;;;
    (e) none of the above.

15. Given the root of a tree as parameter, the following algorithm (scantree) is used to print out the labels of all its tree nodes.

```
Algorithm scantree(treenode root)
  if root =nil return;
  scantree(root.left);
  print root.label;
  Scantree(root.right);
```
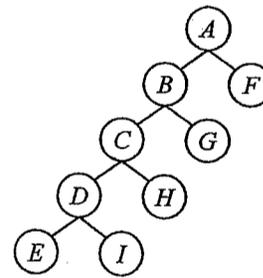

Feeding the root of the above tree as the parameter to the algorithm, what string will be printed out?
    (a) ABFCGDHEI;
    (b) ABCDEFGHI;
    (c) ABCDEIHGF;
    (d) EDICHBGAF;
    (e) none of the above.